Multifractal description of road traffic structure

Jacques Lévy Véhel, Robert Vojak, Mehdi Danech-Pajouh

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Jacques LEVY-VEHEL
Robert VOJAK

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Description Multifractale de la Structure du Trafic Routier

Jacques LEVY-VEHEL, Robert VOJAK
INRIA
Domaine de Voluceau - Rocquencourt
B.P. 105 - 78153 Le Chesnay CEDEX.

Mehdi DANECH-PAJOUH
INRETS
2, avenue du Général Malleret-Joinville
94114 Arceuil Cedex

Abstract

In this work, we study the structure of road traffic with the help of fractal and multifractal tools. Using classical models of traffic that lead to a Burgers' equation, and recent results on the solutions of this equation when the initial conditions are scaling, we predict that, under some circumstances, the traffic can possess a multifractal structure similar to those of multiplicative processes. We then verify this behavior on six minute data of traffic flows. The high sampling rate allows to evidence the highly irregular nature of the flows, and to quantify this irregularity using the classical tools of the multifractal theory, namely the $(q, \tau(q))$ and the $(\alpha, f(\alpha))$ curves. These characterizations in turn permit to classify the complex traffic data, with some application to short-term prediction.

Keywords : traffic, Burgers' equation, multifractal, classification.

Résumé

Dans cet article, nous étudions la structure du trafic routier à l'aide d'outils fractals et multifractals. En utilisant des modèles classiques de trafic qui mènent à une équation de Burgers, et d'après de récents travaux portant sur les solutions de cette équation lorsque les conditions initiales sont self-similaires, nous prédisons que, sous certaines circonstances, le trafic possède une structure multifractale identique à celle de processus multiplicatifs. Nous vérifions alors ce comportement sur des données six minutes de débit de trafic routier. Le taux d'échantillonnage élevé met en évidence la nature très irrégulière des flux, et permet de quantifier cette irrégularité en utilisant les outils classiques de la théorie multifractale, i.e. les courbes des $(q, \tau(q))$ et des $(\alpha, f(\alpha))$. Ces caractérisations permettent à leur tour de classifier les données complexes de trafic, ouvrant des perspectives pour la prédiction à court terme.

Mots-clés : trafic, équation de Burgers, multifractal, classification.

1 Introduction

In the last decades, the growth of road traffic has become more and more important, yielding enormous energy and time wasting, noise, pollution and accidents, and giving rise to health and financial problems.
The necessity of a good understanding of the traffic is thus essential, at all scales. It concerns people stuck in a heavy traffic as well as the society in general.

The issue of this understanding is to create tools necessary to traffic regulation and to the elaboration of transport infrastructures.
As pointed out in Haberman[6], "the ultimate aim is to understand traffic phenomena in order to eventually make decisions which may alleviate congestion, maximize flow of traffic, eliminate accidents, minimize pollution, and other desirable ends."
In this work, we propose to study traffic flows along unidirectional roads. In the following section, we recall a classical approach of dynamic modeling that derives a partial differential equation for traffic flow. This equation is a Burger's equation, and we use recent results on its solutions when the initial conditions are scaling to show that the traffic flows can possess a multifractal structure under some circumstances. Basic recalls on multifractal analysis are given in section 3.
Section 4 is devoted to an empirical study whose goal is to verify the multifractal properties of the traffic data when measured at a sufficiently high sampling rate.
These properties allow to classify the highly complex signal of traffic flows, and give some perspectives for solving the difficult problem of nonlinear traffic prediction.

2 Modeling of Highway Traffic

Throughout this section, we assume that cars move down a long single-lane highway from left to right, between position $x = a$ and $x = b$, at a variable speed, and that they are neither created nor destroyed on our road except through entrances and exits.

Let u be the velocity of any car, ρ the density of the traffic, i.e. the number of cars per unit distance, and q the flow of the traffic, i.e. the number of cars crossing a road section per unit time.

On the basis of actual traffic observations, u can be assumed to depend only on ρ (see [1] and [5]).
Let $u_{max} = u(0)$ be the maximum speed, or *maximum free speed*, and ρ_{max} the maximum possible density at which there is a bumper-to-bumper traffic ($u(\rho_{max}) = 0$).
A reasonable model for $u(\rho)$ used in [6] and [7] is

$$u(\rho) = u_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \tag{1}$$

It is shown in [6] that, if u and ρ depend only on position x and time t, the flow q is

$$q(x,t) = u(x,t)\rho(x,t) \tag{2}$$

Furthermore, we have (see [6] and [7])

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = q(a,t) - q(b,t) \tag{3}$$

which expresses the fact that changes in the number of cars are due only to the flow across the boundary. Equation (3) is an *integral conservation law*, which can also be rewritten

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{4}$$

using (2). Equation (4) is called *conservation of cars.*
Unfortunately, this approach is not satisfactory, as the following example will show : suppose that there is an observer at some location $x_0 = x(0)$ on the highway, and that he moves at some constant velocity.
If we note $x(t)$ his position at time t, then

$$\rho(x(t),t) = \rho(x_0,0) = \rho_0 \tag{5}$$

Therefore the density ρ remains constant along some curves (depending upon x_0) in the x,t plane, whose components are $(x(t),t)$. Those curves are the *characteristics* of the partial differential equation.
It was shown (see [6] and [7]) via a straightforward calculus that

$$x(t) = \frac{dq(\rho_0)}{d\rho}t + x_0 \tag{6}$$

along the characteristic on which $\rho = \rho_0$.
Since x_0 can vary, there is a family of straight line characteristics in the x,t plane on each of which ρ remains constant.
For instance, if we are given

$$\rho_0(x) = \rho(x,0) = \begin{cases} \frac{\rho_{max}}{4} & x < 0 \\ \rho_{max} & x > 0 \end{cases} \tag{7}$$

one has

$$\frac{dq(\rho_0)}{d\rho} = \begin{cases} \frac{u_{max}}{2} & \rho = \frac{\rho_{max}}{4} \\ -u_{max} & \rho = \rho_{max} \end{cases} \tag{8}$$

The corresponding characteristics are given by

$$x(t) = \begin{cases} \frac{u_{max}}{2}t + x_0 & x_0 < 0 \\ -u_{max}t + x_0 & x_0 > 0 \end{cases} \tag{9}$$

and sketched in figure 1.
One can notice that those lines intersect, which means that at some points, the density ρ has two values, and is thereby an unacceptable solution.

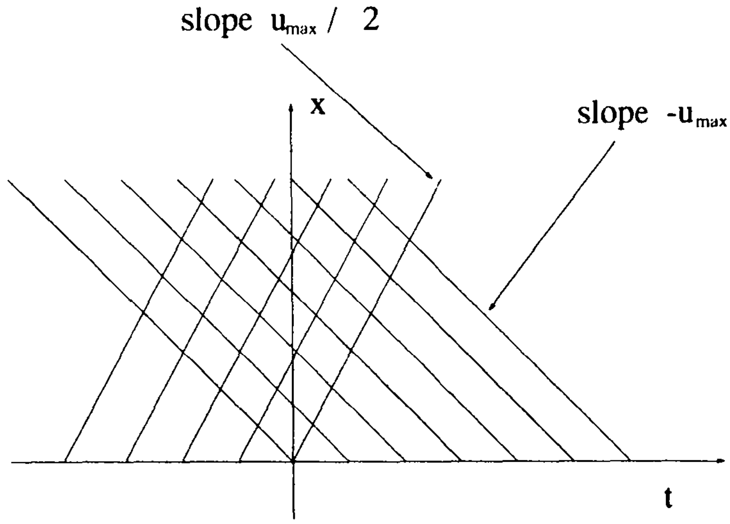

Figure 1: Characteristic curves

An alternate approach (see [6] and [7]) is to replace q with

$$\hat{q} = q - \nu \frac{\partial \rho}{\partial x} \tag{10}$$

(ν is a positive real number), taking in consideration that a driver can adjust his velocity not only according to the density (space between cars), but also to the rate at which this density is changing. Thus u becomes

$$\hat{u} = \frac{\hat{q}}{\rho} = u - \frac{\nu}{\rho} \frac{\partial \rho}{\partial x} \tag{11}$$

Combining (1), (2), (4) and (11) yields

$$\frac{\partial \rho}{\partial t} + u_{max} \left(1 - 2\frac{\rho}{\rho_{max}}\right) \frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \tag{12}$$

or equivalently

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2} \tag{13}$$

with $\psi = u_{max}(1 - 2\rho/\rho_{max})$. Equation (13) is a *Burger's equation.*

Recently, some authors have shown that, under certain conditions, the solutions of Burger's equation possess a fractal structure (see [8] and [11]).
In general, when the initial density $\rho_0(x)$ is smooth, we know that, using a Hopf-Cole transform to solve Burger's equation, after some time and in the inviscid limit ($\nu \to 0$), the

solution $\rho(x,t)$ is smooth except for isolated discontinuities, called *shocks*. For long times, the smooth zones become ramps of slope $1/t$.
When $\rho_0(x)$ is a fractional Brownian motion of scaling exponent h, $h \in]0,1[$, the behavior of the solutions is very different. The self-similarity of the initial conditions has some drastic consequences on the solution. Intuitively, for smooth initial conditions, it takes some times for shocks to be created; as for scaling initial conditions, shocks may appear after arbitrarily short times.
If we denote by $x(a,t)$ the Lagrangian function, i.e. the location at time t of a vehicle initially at position a, the function $a(x,t)$ is called the *inverse Lagrangian function.*
Then, for initial conditions that are fractional Brownian motions of exponent h, it can be shown that (see [8] and [11]):

- the inverse Lagrangian function $a(x)$ for the solution of Burger's equation is a Devil's staircase.
- the total number of shocks per unit length is infinite.
- the probability $P(\Delta a)$ to find a shock interval Δa in an interval $[a_k, a_{k+1}[$, where $a_k = 2^{-k}$ is given by $P(\Delta a) \sim (\Delta a)^s$ when $\Delta a \to 0$ with $s = -h$.
- the set of Lagrangian regular points (i.e. points a that have not participated in any shock during the interval $[0,t[)$ has Hausdorff dimension h, for initial condition with $h = 1/2$. It is conjectured that this result holds for any $h \in]0,1[$.

Now the question for our study of highway traffic is : what is the structure of the initial density $\rho_0(x)$? Is it a smooth function, or does it look like a (fractional) Brownian motion ?

It is difficult to obtain data that would allow to give a precise answer to this question : indeed, what we need, is to have access, on a certain portion of the road, to the instantaneous density $\rho_0(x)$ on a sufficiently large number of points. This implies to be able to put a great number of captors on the road, which is not usually done. In general, there are too few sampling points, and no quantitative statements can be made. However, the following can be said :

- if, inside a given zone, the traffic has not reached at any point the critical density ρ_c (ρ_c is the density such that, for $\rho > \rho_c$, q becomes a decreasing function of ρ), then the initial density $\rho_0(x)$ will probably be a smooth function. However, this situation is not of interest, since it does not raise any serious problems, neither for traffic regulation, nor for prediction.
- if some congestion points exist in the considered portion, due to crashes, road narrowing or any other reason, then $\rho_0(x)$ will present irregularities at many more points than only the congestion ones : indeed, if a car somewhere on the road suddenly slows down or has another irregular behavior, it will have consequences on a large part of the road behind and over a long period of time ; this suggests that irregularities have long range correlation both in time and space. At a certain fixed time t_0, the values of ρ_0 along the x axis reflect all the interactions between irregularities that occurred before t_0.

This picture is reminiscent of what happens in multiplicative processes or instable growth, where, as soon as some small instabilities appear, they develop into larger irregularities.
Of course, in the case of road traffic, a regulation process stops this growth at some time (except in a number of science fiction short stories).
These simple considerations are in no case sufficient to prove that the initial velocity, and by consequence, the solution to Burger's equation for road traffic has a fractal structure. They only indicate that it could be the case, through a mechanism of a multiplicative process or a non-linear growth.
Another way to approach this issue is to use flow records. Indeed, if we do not have access to the instantaneous values of $\rho_0(x)$, it is much easier to record the fluctuations of $q(t)$ at a certain point with a good precision (6 minute data in our case). On the basis of this data, we can compute some experimental quantities to verify the possibly (multi-)fractal structure of road traffic. This we do in the section 4, showing experimental evidence of the multifractality of flow records. Before that, we briefly recall basics of the multifractal theory.

3 Recalls on the Multifractal Theory

We briefly recall some basic facts about the multifractal theory. See also [13], [14], [15], [17], [19], [12].

Let μ be a Borel probability measure on $[0, 1]$. Let ν_n be an increasing sequence of positive integer, and define:

$$I_{i,n} = \left[\frac{i}{\nu_n}, \frac{i+1}{\nu_n}\right]$$

We consider the following quantities:

$$\tau_n(q) = -\frac{1}{\log \nu_n} \log \sum_{0 \le i < \nu_n}^{\star} \mu(I_{i,n})^q$$

where $\sum^\star$ means that the summation runs through those indices i such that $\mu(I_{i,n}) \neq 0$.

We shall say that μ has a multifractal behavior if :

$$\lim_{n\to\infty} \tau_n(q) = \tau(q)$$

exists for q in a non empty interval of $\mathbb{R}$.
$\tau(q)$ characterizes the global behavior of the measure when the size of the intervals tends to zero. $\tau(q)$ is related to a notion of generalized dimensions. Indeed, if we define:

$$\begin{aligned} D_q &= \frac{1}{q-1}\tau(q) \qquad q \neq 1 \\ D_1 &= \lim_{q\to 1}\left[\frac{1}{q-1}\tau(q)\right] \end{aligned}$$

then D_0 is the fractal dimension of the support of μ, D_1 is the information dimension, D_2 the correlation dimension, etc ...
Set:

$$I_n(x) = \{I_{i,n} / x \in I_{i,n}\}$$

We define :

$$E_\alpha = \left\{ x \in [0,1[/ \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log \nu_n} = -\alpha \right\}$$

The exponents α characterize the local scaling behavior of the measure: if α exists at point x then we have:

$$\mu(I_n(x)) \sim (\nu_n)^{-\alpha(x)} \text{ when } n \to \infty$$

ν_n being the linear size of the interval around x upon which we evaluate μ. α is called the Hölder exponent at point x.

E_α can then be seen as the subset of points having the same scaling behavior, described by α.

To have a multifractal description of μ, one first compute the set of possible α exponents, and then evaluate the "size" of the subset E_α of $[0,1[$ associated with α, by computing the Hausdorff dimension of E_α, often denoted by $f(\alpha)$.

This $(\alpha, f(\alpha))$ description is thus both local (via α) and global (via $f(\alpha)$). It is called the multifractal spectrum of μ. Several interpretation of these quantities can be made. One of the most important is the link between $f(\alpha)$ and the rate function appearing in the theory of large deviations. Briefly, this means that the exponential of $f(\alpha) - D$ measures the decay of the probability of finding the value α when n tends to infinity, where D is the dimension of the euclidean embedding space.

A central concern in the multifractal theory is to link both descriptions, namely $(\alpha, f(\alpha))$ and $(q, \tau(q))$. This has important applications. Indeed, $\tau(q)$ is usually much easier to compute on experimental data than $(\alpha, f(\alpha))$: $\tau(q)$ is obtained by averaging over many intervals and then taking the limit. α is more sensitive to noise, since it is computed independently at each point. As for $f(\alpha)$, it implies the computation of a Hausdorff dimension, which is typically very involved.

Under very general assumptions, it has been proven that (see [12]):

$$f(\alpha) \leq \inf_q (q\alpha - \tau(q))$$

For certain special classes of measures, including multinomial measures, we have an equality:

$$f(\alpha) = \inf_q (q\alpha - \tau(q))$$

That is, the Hausdorff dimension of E_α is obtained through a Legendre transform of $\tau(q)$.

In this case, an analogy with thermodynamics can be made, with the following equivalences :

q	$\frac{1}{T}$	$T =$ temperature
α	U	
f	S	
$\frac{\tau}{q}$	$F = U - TS$	

In the case of multinomial measures, $f(\alpha)$ is a bell-shaped curve. This shape is also observed for a number of natural phenomena. However, this is by no way a general property, as one can prove that any ruled function can be the spectrum of a multifractal measure (see [3]).

Other "special" features of f may appear depending on the construction of the measure, as for instance negative values (see [19]).

4 Experiments

4.1 Experimental evidence of the multifractal structure of Highway Traffic

We have computed both the $(q, \tau(q))$and $(\alpha, f(\alpha))$ curves on experimental traffic data.
These data give the flow of the traffic on a highway, measured every 6 minutes, during a continuous period of one month (July 1992).
Figure 2 displays the raw data over one day. One can notice that the signal possesses irregularities at all scales. Figure 3 displays the $(q, \tau(q))$ curve, and figure 4 displays the $(\alpha, f(\alpha))$ spectrum.
Computations made on other days and other time duration (2 hours - 2 days) typically exhibit the same multifractal behavior, namely :

1. the $\tau(q)$ curve clearly departs from a straight line.

2. the spectrum displays the commonly observed bell shape.

The regression lines for the computation of the different quantities fit the data points to within an error always less that 3%.

4.2 Classification of highway traffic data

For various purposes, as for instance the description of the structure of the traffic over a certain period, or the short term prediction, it is desirable to be able to classify the flow curves, according to certain criteria.
For instance, a classical approach to short term prediction (see [2]) is the following one :

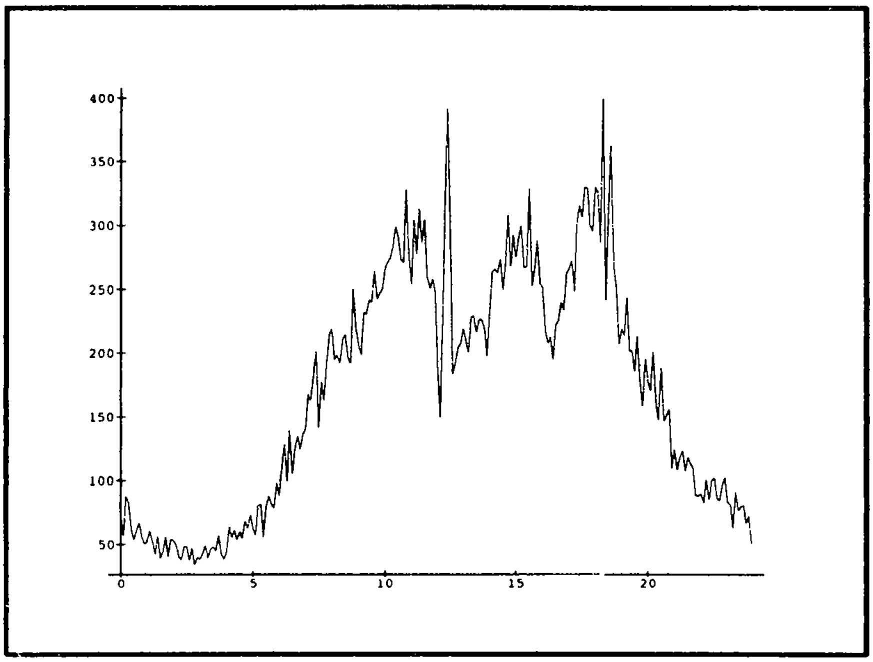

Figure 2: Flow versus time (in hours)

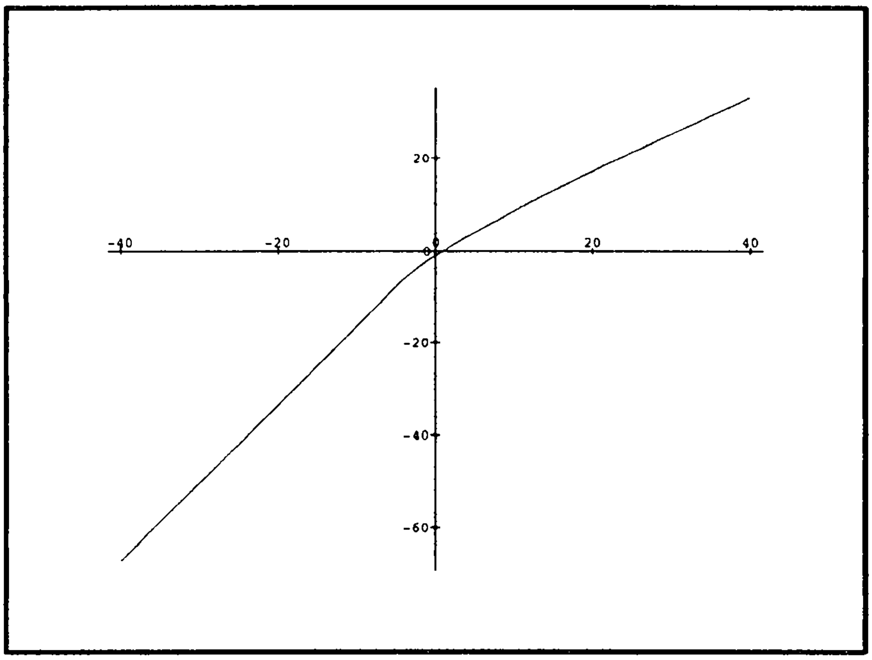

Figure 3: Graph of $\tau(q)$ versus q

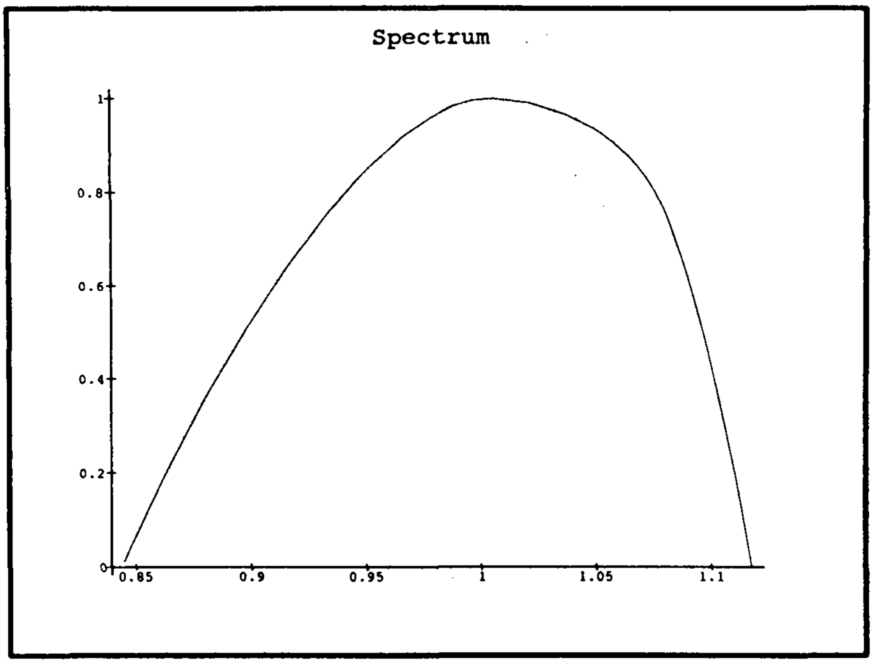

Figure 4: Multifractal Spectrum

- in the first step, observation and processing of historical data is done : from the raw data of traffic flow on a given period of time (for instance the 10 hours preceding the studied time) we compute normalized values by dividing the local flow by the global flow on the period. This allows to get rid of seasonal variations. The obtained vectors (which have, for instance, 20 components in the case of 30 minutes data on 10 hours periods) are called profiles. All the days are then classified, on the basis of the profile data, using a classical data analysis method, such as a cluster algorithm. Typically, 4 classes are defined, grouping homogeneous days with respect to the profiles.
- the second step consists in modeling each class with a simple autoregressive model.
- in the third step, one perform the short term prediction itself ; one identifies the class to which the data to be analysed is supposed to belong, and applies the corresponding autoregressive model to it.

In this section, we propose a classification scheme that takes into account the fractal structure of the data evidenced in the previous section.

A classification based on the multifractal properties of the signal should hopefully lead, in a certain fashion, to a more pertinent analysis and thus to better short term predictions. Indeed, if one wants to make robust forecastings, the first important thing to do is to identify the structure of the process : if it is approximately linear, then a linear method (for instance an autoregressive one) can be helpful. On the contrary, if it is (multi-)fractal, completely different types of models should be used. One should further be able to distinguish between several non linear behaviors, since it is likely that, the more irregular the process is, the more "non-linear" the methods should be. In other words, one has to quantify by how much the process departs from a linear one, in order to decide whether linear methods with small corrections can be used, or what sort of more complex models are needed.

The computations of the multifractal spectrum or of the $\tau(q)$ curve can help this purpose. We proceed as follows :

- We first have to decide what we are going to classify. We use as basic patterns "slices" of the data that are the flows recorded between times t_1 and t_2. The size of the interval $[t_1, t_2]$ depends partly on the prediction time, and partly on the structure of the data : if we intend to characterize precisely what happens at one moment, we should use small intervals, but then the classification might be too sensitive to noise or specially rare events. Thus a trade-off has to be found. We tried periods of 2, 3, and 4 hours.
- Once we have sliced our data into m periods of n hours, we compute $\tau(q)$ for each slice.
- We finally classify all the $\tau(q)$ curves according to some criterion, in order to obtain some types of average behaviors of the signals with respect to their singularities

On figure 5 are presented the classes obtained with a classical classification method applied directly to the flows (two hour and three hour data). Four classes are detected, with well balanced numbers of samples (in this case). On figure 6 are the classes found with a computation on the $\tau(q)$. The situation here is very different. We have five very different classes :

1. class 1 contains a large number of samples (294 for two hour data, 179 for three hour data). The corresponding $\tau(q)$ curve is almost a straight line, indicating a monofractal behavior (the slope does not equal one). The interpretation is that most two-hours slices have a behavior characterized by a single exponent, and can be studied as such. Thus, most of the time, the traffic flow (under the conditions of the classification) can be seen as a homogeneous irregular signal.
2. class 2 and 3 contain respectively 35 and 32 samples for two hour data, and 19 and 28 for three hour data. The two corresponding $\tau(q)$ curves clearly show a multifractal behavior.
3. class 4 and 5 contain very few samples (2 and 5 for two hour data, 13 and 8 for three hour data), and also exhibit multifractal behavior, with a more non linear shape for $\tau(q)$.

One can then use these results in the following fashion : when a new two-hour sample is presented, we find to which class it belongs, by computing its $\tau(q)$. If class 1 is found (we know that it will be the case most of the time), a model based on a monofractal behavior may be applied. Sometimes, the situation will be worse, as the sample will belong to class 2 or 3, and a more sophisticated model will have to be used. At last, very rarely, an even more irregular behavior will occur, characterized by class 4 and 5.

It is interesting to go back to the samples, and see what the multifractal classification looks like in the flows space instead of the $(q, \tau(q))$ one. We have drawn on figure 7 the mean flow curves corresponding to the $\tau(q)$ classes of figure 6. We verify that class 1 (flat curve) corresponds to a much more regular behavior than the other classes. Besides, the fact that the five classes have clearly different shapes is a sign of good separation of the different possible structures, and thus of the ability of the method to be used as a first step in a prediction scheme.

The difference with the classical classification is obvious on this figures : instead of classifying the shapes of the profiles, the multifractal approach highlights the different types of singular behavior, each class being characterized by the amount of irregularities it contains.

4.3 Short time prediction

In this section, we do no more than indicate some ideas for prediction in the case of a multifractal signal.

The prediction of chaotic signals have been considered by a number of authors (see [10]). Specially interesting is the use of chaos theory to find an appropriate space (more precisely an appropriate space dimension) in which one can do the prediction, using local approximation methods.

Here we outline the possible use of the multifractal spectrum for performing such a task. The $(\alpha, f(\alpha))$ spectrum gives us informations about both the local (via α) and global (via $f(\alpha)$) behavior of the signal (see figure 4).

Suppose that we want to predict what is going to happen in the very near future of time t. Using the spectrum, we know what range of exponents α can occur at t, and the probability that a particular exponent will appear, which is related to $f(\alpha)$: the mean behavior is characterized by α_0 which corresponds to the maximum value of $f(\alpha)$, and the probability of occurrence of other α values is proportional to the exponential of $f(\alpha) - 1$ (see for instance [12] or [9]).

Now, by definition of the local exponent α, the jump of the flow during a given small period of time δt is of order of $(\delta t)^\alpha$ (in fact, we may be a little more precise, see [4]). Thus we can predict that the flow at time $t + \delta t$ will be in a cone, of vertex the flow at time t and whose angle is related to α, with a probability related to $f(\alpha)$. Depending on the multifractal spectrum (and specially on its width and bandpass), more or less precise predictions can be made using this geometrical approach.

Obviously these are only a few preliminary first steps towards prediction for multifractal signals. Much more work remains to be done in order to obtain a coherent method for solving this difficult problem.

5 Conclusion

We have studied from a multifractal point of view the structure of road traffic. Theoretical models, leading to a Burger's equation, allow to understand how a fractal behavior can appear.

Besides, calculations conducted on experimental data evidence a multifractal structure for the traffic flows. The quantification of these irregularities is a first step towards classification

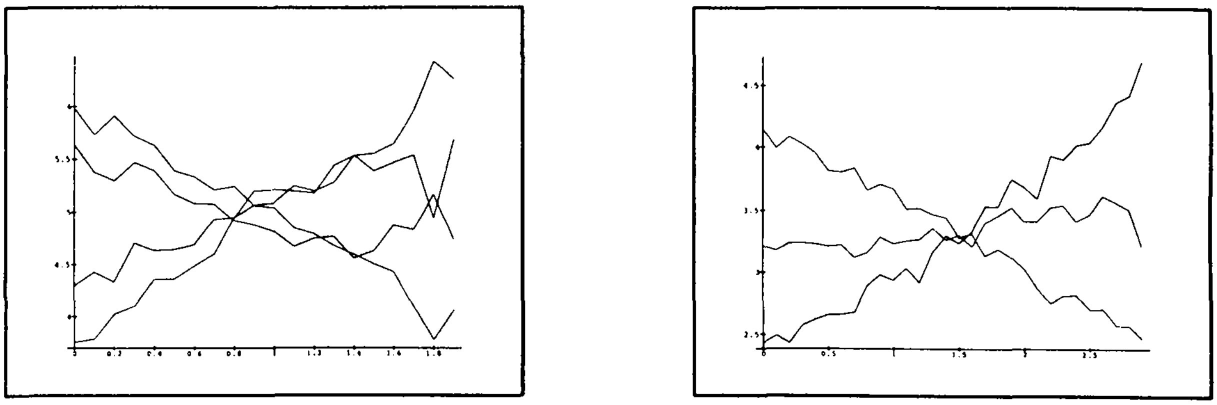

Figure 5: Results of the classification of the profiles using the raw 2 hour (left) and 3 hour (right) data

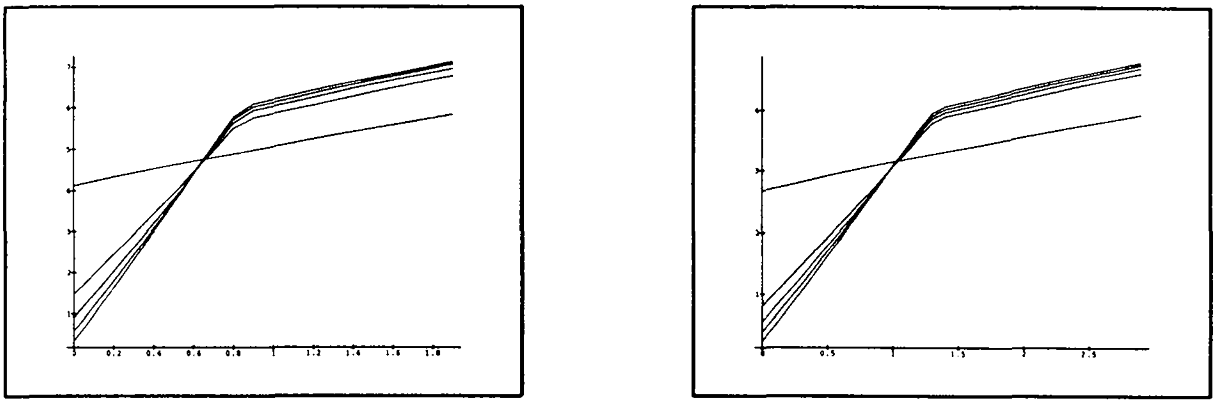

Figure 6: Results of the classification of the $\tau(q)$ curves for the 2 hour (left) and 3 hour (right) data

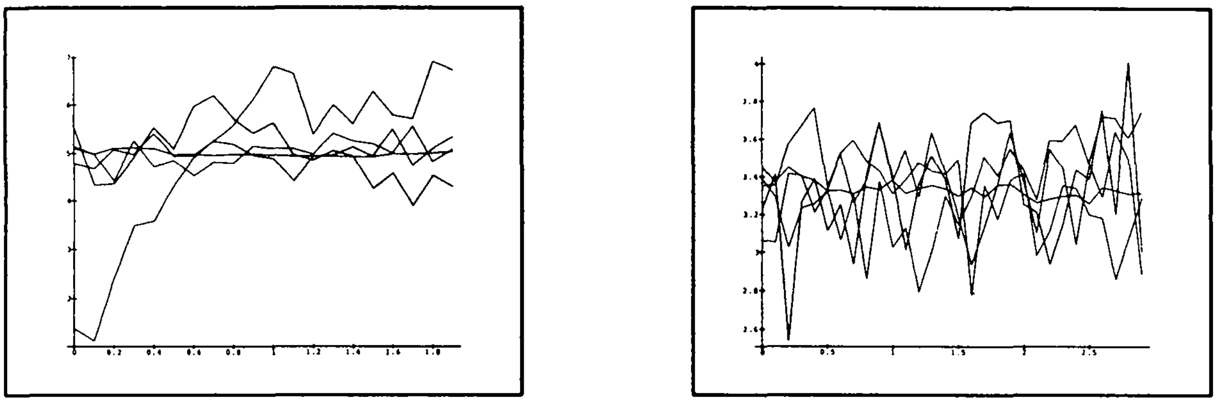

Figure 7: Results of the classification of the profiles using the multifractal characterization of the 2 hour (left) and 3 hour (right) data

and prediction of complex road traffic signal. However, much more work is needed to fully understand the various phenomena, and thus be able to give more robust predictions.

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Unité de Recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)

Unité de Recherche INRIA Lorraine Technopôle de Nancy-Brabois - Campus Scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 VILLERS LES NANCY Cedex (France)
Unité de Recherche INRIA Rennes IRISA, Campus Universitaire de Beaulieu 35042 RENNES Cedex (France)
Unité de Recherche INRIA Rhône-Alpes 46, avenue Félix Viallet - 38031 GRENOBLE Cedex (France)
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